A Solution to Functional Tetrahedron Equation with Twisted Cross–Ratios

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1. A Functional Transformation for Edge Variables from Refactorization Equation

Consider the following "refactorization equation" for the product of three matrices:

\[
\begin{pmatrix}
  a_1 & b_1 & 0 \\
  c_1 & d_1 & 0 \\
  0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
  a_2 & 0 & b_2 \\
  0 & a_3 & b_3 \\
  0 & c_2 & d_2
\end{pmatrix} = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & a'_3 & b'_3 \\
  0 & c'_2 & d'_2
\end{pmatrix} \begin{pmatrix}
  a'_2 & 0 & b'_2 \\
  0 & 1 & 0 \\
  c'_1 & d'_1 & 0
\end{pmatrix},
\]

(a_1, \ldots, d'_3 are numbers) for the case when all six submatrices \( \begin{pmatrix} a_i^{(t)} \\ b_i^{(t)} \end{pmatrix} \) have the form

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} = \begin{pmatrix}
  \alpha & 1-\alpha \\
  1-\beta & \beta
\end{pmatrix}.
\]

In other words, each of the six matrices in (1) transforms the vector \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) into itself.

It is known from [1, 2, 3] that each side of (1) determines the other side to within some "gauge freedom", and one can verify that the additional conditions (2) are exactly good for fixing that freedom.

The fate of an arbitrary vector \( \begin{pmatrix} p \\ q \\ r \end{pmatrix} \) under the action of both sides of (1) is more complicated. We present it in fig. 1, where we denote the matrices entering (1), in their order in that equation, by letters \( X_1, X_2, X_3, Y_3, Y_2, Y_1 \).

The meaning of the LHS of Figure 1 is that

\[
X_3 \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} p \\ v \\ w \end{pmatrix}, \quad X_2 \begin{pmatrix} p \\ v \\ w \end{pmatrix} = \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \quad X_1 \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},
\]
while the meaning of the RHS is that

\[
\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} f \\ g \\ r \end{pmatrix}, \quad \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\]

One can see that if, vice versa, all the values \( x, y, z, \ldots \) in e.g. the LHS of fig. 1 are given, then matrices \( X_1, X_2, X_3 \) of the form (2) are recovered unambiguously. So, we can take some given values of nine numbers in the LHS, get the triple of matrices \( X_1, X_2, X_3 \) from them, then get \( Y_1, Y_2, Y_3 \) by (1), and then get the missing values \( f, g, h \) in the RHS from \( p, q, r \) using \( Y_1, Y_2, Y_3 \). We will formulate this the following way: for any fixed “outer” variables \( x, y, z, p, q, r \), the transformation

\[
R = R(x, y, z, p, q, r): (u, v, w) \mapsto (f, g, h)
\]

is given.

The transformations (3) satisfy the functional tetrahedron equation (FTE). To explain this, note that equation (1) can be naturally regarded as an equation in the direct sum of three one-dimensional complex linear spaces, each of the matrices acting nontrivially only in a direct sum of two of them. One can consider similar relations in a direct sum of four spaces (each of the matrices acting nontrivially again only in a direct sum of two spaces). Let us picture in fig. 2 the spaces as straight lines, put matrices at their intersections, and attach the results of matrix action upon some 4-vector to line segments like in fig. 1, and then consider the transition from the LHS of fig. 2 to its RHS as a composition of “elementary” transformations \( R \) of type (3).
As was explained in the paper [3] (and the reader will verify it himself easily), there exist two different compositions of four $R$’s both transforming the LHS of fig. 1 in its RHS. The first of them starts with $R_{356}$, by which we mean “turning inside out” triangle 356, while the other—with $R_{123}$. We can write FTE in the same abstract form as in [3]:

$$R_{123} \circ R_{145} \circ R_{246} \circ R_{356} = R_{356} \circ R_{246} \circ R_{145} \circ R_{123},$$

but the sense of (4) is now different: $R$ is now a transformation of variables belonging to the edges rather than of matrices belonging to vertices.

To prove FTE (4) for edge variables, note that the variables belonging to inner edges (i.e., say, edges 12, 13, ..., 56 in the LHS of fig. 2) are unambiguously recovered if variables at outer edges and matrices at vertices are given. The FTE for matrices, according to [3], does hold, while the variables at outer edges are not changed by the transformations. Thus, the variables at inner edges do not depend on the way of transformations as well.


Let us now vary the edge variables in fig. 1, with matrices $X_1, ..., Y_3$ fixed. For instance, consider the variables at outer edges of the LHS of that Figure as functions of three inner variables $u, v, w$, and calculate the corresponding partial derivatives. The reader will easily check that

$$\frac{\partial x}{\partial u} = \frac{x - v}{u - v}, \quad \frac{\partial x}{\partial v} = \frac{x - u}{v - u}, \quad \frac{\partial y}{\partial u} = \frac{y - v}{u - v}$$

and so on.

Using formulae of the type (5), it is not hard to obtain the following relations for “volume elements”:

$$\text{dx} \wedge \text{dy} \wedge \text{dz} = \frac{x - y}{u - v} \cdot \frac{z - u}{w - u} \cdot \text{du} \wedge \text{dv} \wedge \text{dw}$$

from the LHS of fig. 1 and similarly

$$\text{dx} \wedge \text{dy} \wedge \text{dz} = \frac{x - h}{f - h} \cdot \frac{y - z}{g - h} \cdot \text{df} \wedge \text{dg} \wedge \text{dh}$$

from its RHS. The equalness of the RHSs of (6) and (7) can be called “the relation between $\text{du} \wedge \text{dv} \wedge \text{dw}$ and $\text{df} \wedge \text{dg} \wedge \text{dh}$ got via $\text{dx} \wedge \text{dy} \wedge \text{dz}$”.

Similarly, the equalness of the RHSs of relations

$$\text{dy} \wedge \text{dz} \wedge \text{dp} = \frac{y - u}{v - u} \cdot \frac{z - p}{w - u} \cdot \text{du} \wedge \text{dv} \wedge \text{dw}$$

(8)

and

$$\text{dy} \wedge \text{dz} \wedge \text{dp} = \frac{y - z}{g - h} \cdot \frac{p - g}{f - g} \cdot \text{du} \wedge \text{dv} \wedge \text{dw}$$

(9)

can be called “the relation between $\text{du} \wedge \text{dv} \wedge \text{dw}$ and $\text{df} \wedge \text{dg} \wedge \text{dh}$ got via $\text{dy} \wedge \text{dz} \wedge \text{dp}$”. There are four more pairs of relations of the type (6—9) with $\text{dz} \wedge \text{dp} \wedge \text{dq}, \text{dp} \wedge \text{dq} \wedge \text{dr}, \text{dq} \wedge \text{dr} \wedge \text{dx}$ and $\text{dr} \wedge \text{dx} \wedge \text{dy}$ respectively in their LHSs.

Certainly, one can exclude the differentials from those relations and obtain formulæ giving explicitly the connection between edge variables, i.e. the transformation $R$, namely

$$\frac{x - y}{u - y} \cdot \frac{u - z}{p - z} = \frac{x - h}{f - h} \cdot \frac{f - g}{p - g},$$

(10)
\[
\begin{align*}
y - x \cdot v - r &= y - h \cdot g - f, \\
v - x \cdot q - r &= g - h \cdot q - f, \\
z - p \cdot w - q &= z - g \cdot h - f, \\
w - p \cdot r - q &= h - g \cdot r - f, \\
x - v \cdot u - w &= x - r \cdot f - q, \\
u - v \cdot p - w &= f - r \cdot p - q, \\
y - u \cdot v - w &= y - z \cdot g - p, \\
v - u \cdot q - w &= g - z \cdot q - p, \\
z - u \cdot w - v &= z - y \cdot h - x, \\
w - u \cdot r - v &= h - y \cdot r - x.
\end{align*}
\]

Note that if we set \( p = x \) in (10) or (13) then each of these equations becomes just an equality of two cross–ratios. Similarly, equations (11) and (14) become such equalities if we set \( y = q \), while (12) and (15) become equalities of such kind if we set \( z = r \). So, we call the relations (10)—(15) equalities with twisted cross–ratios.

The relation
\[
\frac{x - y \cdot z - u}{u - v \cdot w - u} \, du \wedge dv \wedge dw = \frac{x - h \cdot y - z}{f - h \cdot g - h} \, df \wedge dg \wedge dh
\]

that follows from (6) and (7) can be viewed as a kind of duality for two scattering amplitudes corresponding to the LHS and RHS of fig. 1. More generally, we can consider relation (16) together with relations (10)—(15) raised in arbitrary degrees (the relations (10)—(15) are not independent, so one of those degrees can be set to zero; the relation (16) is taken, of course, in degree 1). Then we can multiply separately the LHSs and RHSs of all these relations. What we get looks very much like as the duality for Veneziano amplitude, but now the \( s \leftrightarrow t \) duality is replaced with the “Yang–Baxter duality” of fig. 1, and thus one differential is replaced by a wedge product of three differentials in each side of the relation.

Conclusions

We have thus presented a new solution to the functional tetrahedron equation, given by formulas (10)—(15). Note the interest in such solutions expressed in the recent paper [4] devoted to classification of discrete integrable equations. Moreover, we have found an interesting differential duality (formula (16) together with (10)—(15)) which corresponds to the functional tetrahedron equation in the same sense as the \( s \leftrightarrow t \) duality corresponds to the functional pentagon equation. This seems to be some new kind of duality whose significance for mathematics and physics is still to be revealed.

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References